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A GENERALIZED FORCE MEASURE OF CONDITIONS AT CRACK TIPS

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Abstract—A tentative measure of the forces tending to cause crack growth—the apparent crack extension force—is defined within the framework of continuum mechanics. By an associated fracture criterion initiation of growth may be predicted as well as the direction of preferred growth. The theory is specialized to elastic, viscoelastic and elastic–plastic materials. Under specified conditions the apparent crack extension force may be expressed by surface integrals over the boundary of an arbitrary part of the body for quasi-static deformation and for steady-state propagation of the crack. For plane deformation and for infinitesimal deformation under plane stress conditions these surface integrals reduce to path independent line integrals which include the J integral by Rice[1] and the G integral by Sih[2] as special cases.

1. INTRODUCTION

During the last years much interest has been focused on path independent integrals in fracture mechanics. Rice[1] introduced the J integral as a measure of a crack extension force in elastic materials. It applies to infinitesimal deformation under planar conditions. The J integral generalizes the crack extension force $\mathcal{G}$ introduced by Irwin[3]. It is also a special case of the energy momentum tensor defined by Eshelby[4] as a generalized force acting on an inhomogeneity in an elastic body. The J integral can be used to predict initiation of growth of a plane crack in its own plane.

Sih[2] discussed dynamic aspects of crack propagation and through an approach similar to that of Rice derived the G integral which is valid for steady-state propagation of cracks in elastic materials at small strain.

The objective of this paper is to derive a generalized force measure of the forces tending to cause crack growth in solids. A crack in a continuous body is described by a material singular surface or surface of discontinuity. The motion of particles on that surface may be discontinuous to allow for the crack to extend. With the singular surface a specific surface energy is associated. During extension of the crack, the surface energy is assumed to be supplied by mechanical work in the so called cohesive zone by cohesive forces acting between the separated surfaces. The kinematical properties of the model are formulated in Section 2, being devoted to continuum mechanical preliminaries. In Section 3 the balance of energy for simple thermomechanical materials is investigated. It turns out that for the model considered the concepts of surface energy and cohesive force are equivalent, provided that heat does not supply the surface energy.

A suitable application of the principle of virtual work in Section 4 and an interpretation of the contribution from the cohesive zone suggest the definition of the apparent crack extension force. The apparent crack extension force is expressed by a volume integral over an arbitrary part of the body containing a crack and by a surface integral over the boundary of that part. Introduction of an instantaneous potential renders it possible to transform the volume integral into a surface integral in case of quasi-static deformation or steady-state propagation of the crack. The problem of construction of that potential is discussed in Section 5. With the apparent crack extension force a fracture criterion is associated by which initiation of growth may be predicted as well as the direction of preferred growth. Furthermore, the definition of the apparent crack extension force is formally extended to the case where the crack border is modeled by a singular line.

In Section 6 the apparent crack extension force is specialized to plane deformation and in case of small strain also to plane stress conditions. The resulting expressions include the J integral and the G integral as special cases.

2. CONTINUUM MECHANICAL PRELIMINARIES

A body $\mathscr{B}$ is identified by a reference configuration $\boldsymbol{\kappa}(\mathscr{B})$ in a region $v_R(\mathscr{B})$ of a three-dimensional Euclidean point space. An arbitrary point $\mathbf{X}$ in $v_R(\mathscr{B})$ is referred to as the particle $\mathbf{X}$ of the body. The boundary of $v_R(\mathscr{B})$ is denoted by $\partial v_R(\mathscr{B})$ and its outward unit normal by $\mathbf{n}_R$. The reference configuration is, for convenience, assumed to be a natural state, i.e. the stress corresponding to $\boldsymbol{\kappa}$ is zero. This configuration will also be referred to as the undeformed configuration.

The motion of the body $\mathscr{B}$ is given by

$$\mathbf{x} = \chi_{\boldsymbol{\kappa}}(\mathbf{X}, t), \tag{2.1}$$

where $\mathbf{x}$ is the place in the spatial configuration occupied by the particle $\mathbf{X}$ at time t.

The concept of singular surface, treated extensively in [5], will be employed as a model of a crack in a solid body. The volume $v_R(\mathscr{P})$ of some part $\mathscr{P}$ of the body in the reference configuration is separated into $v_R{}^+$ and $v_R{}^-$ by a surface s_R as shown in Fig. 1. The boundary ∂v_R is separated into $\partial v_R{}^+$ and $\partial v_R{}^-$.

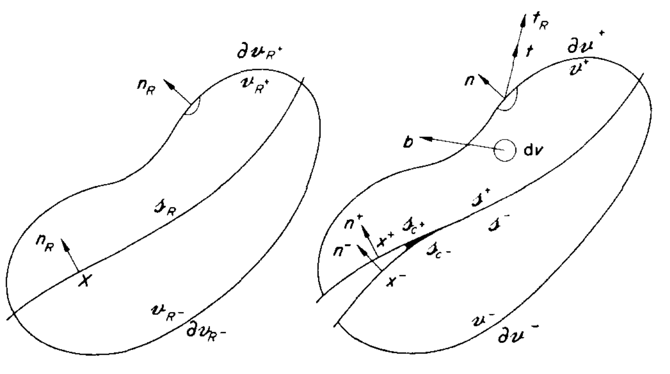

Fig. 1. Reference and spatial configurations of a body containing a crack.

The surface s_R is ass umed to be material, i.e. the surface is described by an equation of the form

$$G(\mathbf{X}) = 0. \tag{2.2}$$

The motion of a particle $\mathbf{X}$ on s_R may be discontinuous and consequently mapped into different points $\mathbf{x}^+$ on s^+ and $\mathbf{x}^-$ on s^-. The unit normal vector $\mathbf{n}_R$ at the particle $\mathbf{X}$ on

$\mathscr{s}_R$ pointing into the v_R^+ region is mapped into $\mathbf{n}^+$ at $\mathbf{x}^+$ and into $\mathbf{n}^-$ at $\mathbf{x}^-$. For particles within v_R^+ and v_R^- as well as for particles on the boundary ∂v_R the motion is continuous.

The one-sided limits of any sufficiently smooth quantity Ψ is denoted by

$$\Psi^{\pm} = \lim_{\mathbf{x}\to\mathbf{x}^{\pm}} \Psi(\mathbf{x}, t) \qquad \text{for } \mathbf{x} \in v^{\pm} \tag{2.3}$$

and the jump of Ψ across the surface is denoted by

$$[\Psi] = \Psi^+ - \Psi^-. \tag{2.4}$$

If $[\Psi] \neq \mathbf{0}$, the surface $\mathscr{s}_R$ is said to be singular with respect to Ψ.

Anticipating the concept of cohesive zone to be defined in Section 3, the definition of the surface $\mathscr{s}_R$ will now be modified. The part of the surface that constitutes the fully opened crack, i.e. the crack outside the cohesive zone, will henceforth be included in the boundary ∂v_R with the normal vector $\mathbf{n}_R$ directed outwards. Accordingly, boundary conditions prescribed on the fully opened crack could be more feasibly expressed. The cohesive part of the crack is denoted by $\mathscr{s}_{cR}$.

The classical fracture mechanical theories may be considered as limit cases where the cohesive zone becomes a singular line propagating in the material, splitting each particle $\mathbf{X}$ in its progress into two particles $\mathbf{X}^+$ and $\mathbf{X}^-$ which are mapped into points $\mathbf{x}^+$ and $\mathbf{x}^-$ respectively.

For simple materials the local motion in the neighbourhood of a given particle is described by the deformation gradient

$$\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{x}, \tag{2.5}$$

where ∇ is the gradient operator.

Continuous motion requires that the positive quantity

$$J = \det \mathbf{F} \tag{2.6}$$

does not approach zero or infinity.

The deformation of a material volume element dv_R in the reference frame into dv in the spatial frame is given by

$$dv = J\, dv_R. \tag{2.7}$$

This equation implies the relationship

$$\rho_R = J\rho \tag{2.8}$$

between the mass densities ρ_R and ρ in the material and spatial configurations respectively.

The deformation of a material vector-area $\mathbf{n}_R\, ds_R$ is

$$\mathbf{n}\, ds = J(\mathbf{F}^{-1})^T\mathbf{n}_R\, ds_R, \tag{2.9}$$

where $(\mathbf{F}^{-1})^T$ is the transposed inverse of $\mathbf{F}$.

The stress vector $\mathbf{t}$ acting upon a material surface with the normal vector $\mathbf{n}$ is given by

$$\mathbf{t} = \mathbf{T}\mathbf{n}, \tag{2.10}$$

where $\mathbf{T}$ is the Cauchy stress tensor.

Introducing the first Piola–Kirchhoff tensor $\mathbf{T}_R$, defined by

$$\mathbf{T}_R = J\mathbf{T}(\mathbf{F}^{-1})^T, \tag{2.11}$$

the stress vector $\mathbf{t}_R$ is given by

$$\mathbf{t}_R = \mathbf{T}_R \mathbf{n}_R, \tag{2.12}$$

where $\mathbf{n}_R$ is the surface normal in the reference configuration. The stress vectors $\mathbf{t}$ and $\mathbf{t}_R$ are related by

$$\mathbf{t}\, ds = \mathbf{t}_R\, ds_R, \tag{2.13}$$

where ds and ds_R are the magnitudes of a material surface element in the spatial and material configurations respectively.

The body is subjected to a field of assigned body forces $\mathbf{b}$ and surface tractions $\mathbf{t}$ (or $\mathbf{t}_R$) on the boundary ∂v.

A body in motion and a corresponding system of forces constitute a dynamical process, if the balance equations for momentum and moment of momentum are satisfied. For regular parts of the body these are equivalent to Cauchy's laws of motion:

$$\begin{aligned} \operatorname{div} \mathbf{T} + \rho \mathbf{b} &= \rho \mathbf{a}, \\ \mathbf{T} &= \mathbf{T}^T, \end{aligned} \tag{2.14}$$

or

$$\begin{aligned} \operatorname{Div} \mathbf{T}_R + \rho_R \mathbf{b} &= \rho_R \mathbf{a}, \\ \mathbf{T}_R \mathbf{F}^T &= \mathbf{F}\mathbf{T}_R{}^T, \end{aligned} \tag{2.15}$$

where div and Div are the divergence operators with respect to $\mathbf{x}$ and $\mathbf{X}$ respectively, and $\mathbf{a} = \ddot{\mathbf{x}}$ is the particle acceleration. The balance of momentum and moment of momentum over the singular surface σ is given by

$$\begin{aligned} [\mathbf{T}_R]\mathbf{n}_R &= \mathbf{0}, \\ [\mathbf{x}] \times \mathbf{t}_R{}^{\pm} &= \mathbf{0}. \end{aligned} \tag{2.16}$$

Since the relationship between the deformed surface elements ds^+ and ds^- is rather complicated, it is not advantageous to use Cauchy's stress tensor in this regard.

Especially for small strains it is convenient to employ the displacement vector $\mathbf{u}$ defined by

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{2.17}$$

in a common frame and the displacement gradient

$$\mathbf{H} = \nabla_{\mathbf{X}} \mathbf{u}. \tag{2.18}$$

The Green–St. Venant strain tensor $\mathbf{E}$ may be defined in terms of the displacement gradient as

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}). \tag{2.19}$$

Omitting the quadratic term, (2.19) reduces to the infinitesimal strain tensor

$$\tilde{\mathbf{E}} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T). \tag{2.20}$$

Finally, we introduce a bracket notation to be used frequently

$$\begin{aligned} \mathbf{A} &= [\mathsf{L}]\mathbf{B} = [\mathbf{B}]\mathsf{L}, \\ \mathbf{A} &= \mathsf{L}[\mathbf{B}] = \mathbf{B}[\mathsf{L}], \end{aligned} \tag{2.21}$$

which should be interpreted according to

$$A_{kl} = L_{ijkl} B^{ij}, \qquad A_{ij} = L_{ijkl} B^{kl}, \tag{2.22}$$

i.e. depending on the position of the bracket, summation should be carried out with respect to each appropriate index consecutively, beginning with the first one, or in inverted order beginning with the last index.

3. BALANCE OF ENERGY OVER COHESIVE ZONES

An arbitrary part $\mathscr{P}$ of a body $\mathscr{B}$ containing a crack represented by a singular surface $\mathscr{s}_R \in v_R(\mathscr{P})$ is considered. With this singular surface and each dynamical process is, following Griffith[6], associated a surface energy Σ defined by

$$\Sigma(\mathscr{P}) = \int_{\mathscr{s}_R(\mathscr{P})} \sigma_R \, \mathrm{d}s_R, \tag{3.1}$$

where σ_R is the specific surface energy referred to the reference configuration and $\mathscr{s}_R(\mathscr{P})$ denotes the part of the surface that is contained in $v_R(\mathscr{P})$.

Including the surface energy, the general balance equation for energy for simple, thermo-mechanical materials takes the form

$$\dot{E} + \dot{\Sigma} + \dot{K} = M + Q. \tag{3.2}$$

Here, E is the internal energy defined by the specific internal energy ε through

$$E(\mathscr{P}) = \int_{v_R(\mathscr{P})} \varepsilon \rho_R \, \mathrm{d}v_R. \tag{3.3}$$

With $\mathbf{v} = \dot{\mathbf{x}}$ denoting particle velocity, the kinetic energy K is

$$K(\mathscr{P}) = \frac{1}{2} \int_{v_R(\mathscr{P})} \mathbf{v} \cdot \mathbf{v} \rho_R \, \mathrm{d}v_R. \tag{3.4}$$

The mechanical power M of the assigned forces is

$$M(\mathscr{P}) = \int_{v_R(\mathscr{P})} \mathbf{v} \cdot \mathbf{b} \rho_R \, \mathrm{d}v_R + \oint_{\partial v_R(\mathscr{P})} \mathbf{v} \cdot \mathbf{t}_R \, \mathrm{d}s_R. \tag{3.5}$$

The heat power Q is defined by

$$Q(\mathscr{P}) = \int_{v(\mathscr{P})} q\rho \, \mathrm{d}v - \oint_{\partial v(\mathscr{P})} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}s = \int_{v_R(\mathscr{P})} q\rho_R \, \mathrm{d}v_R - \oint_{\partial v_R(\mathscr{P})} J(\mathbf{F}^{-1}\mathbf{h}) \cdot \mathbf{n}_R \, \mathrm{d}s_R, \tag{3.6}$$

where q is the specific heat absorption and $\mathbf{h}$ is the heat flux vector.

Change of surface energy is assumed to be restricted to a part of the singular surface $\mathscr{s}_R$ which is called the cohesive zone $\mathscr{s}_{cR}$. Accordingly, substitution of (3.1 and 3.3–3.6) into (3.2) yields

$$\int_{\mathscr{s}_{cR}(\mathscr{P})} \dot{\sigma}_R \, \mathrm{d}s_R + \int_{v_R(\mathscr{P})} (\dot{\varepsilon} - \mathbf{v} \cdot (\mathbf{b} - \mathbf{a}) - q)\rho_R \, \mathrm{d}v_R - \oint_{\partial v_R(\mathscr{P})} (\mathbf{v} \cdot \mathbf{T}_R - J\mathbf{F}^{-1}\mathbf{h}) \cdot \mathbf{n}_R \, \mathrm{d}s_R = 0. \tag{3.7}$$

Green transformation of the surface integral over the boundary ∂v_R yields in view of $(2.15)_1$

$$\int_{v_R(\mathscr{P})} (\dot{\varepsilon}\rho_R - [\mathbf{T}_R]\dot{\mathbf{F}} - q\rho_R + J \operatorname{div} \mathbf{h})\, dv_R + \int_{s_{cR}(\mathscr{P})} (\dot{\sigma}_R - [\mathbf{v} \cdot \mathbf{T}_R - J\mathbf{F}^{-1}\mathbf{h}]) \cdot \mathbf{n}_R\, ds_R = 0. \tag{3.8}$$

Decreasing the volume $v_R(\mathscr{P})$ to zero while $s_{cR}(\mathscr{P})$ is kept constant results in

$$\dot{\Sigma}(\mathscr{P}) = \int_{s_{cR}(\mathscr{P})} \dot{\sigma}_R\, ds_R = \int_{s_{cR}(\mathscr{P})} [\mathbf{v} \cdot \mathbf{T}_R - J\mathbf{F}^{-1}\mathbf{h}] \cdot \mathbf{n}_R\, ds_R \tag{3.9}$$

which is the energy balance over the singular surface.

It is obvious that the surface energy in general may be contributed by heat flux as well as by mechanical work. To proceed, we make the constitutive assumption that the surface energy is not affected by the heat flux $\mathbf{h}$. Thus, (3.9) may be separated into one mechanical and one thermal part. Furthermore, the part $\mathscr{P}$ is arbitrarily chosen. On that account and by aid of (2.16) we conclude that

$$[J\mathbf{F}^{-1}\mathbf{h}] = \mathbf{0} \tag{3.10}$$

and

$$\dot{\sigma}_R = [\mathbf{v}] \cdot \mathbf{T}_R \mathbf{n}_R \tag{3.11}$$

where $\mathbf{T}_R$ is arbitrarily $\mathbf{T}_R^{\ +}$ or $\mathbf{T}_R^{\ -}$.

For the surface energy to increase during crack opening, it is necessary to assume the existence of cohesive forces acting between the separated surfaces in the vicinity of the crack tip. This part of the crack is the cohesive zone s_{cR}. The equation (3.11) expresses in general terms the equivalence of the concepts of surface energy and cohesive force.

4. DEFINITION OF THE APPARENT CRACK EXTENSION FORCE

The principle of virtual work will be applied to an arbitrary part $\mathscr{P}$ of a body containing a crack which is represented by a singular surface with a finite cohesive zone. The appropriate form of the principle of virtual work in absence of internal constraints is

$$\int_{v_R(\mathscr{P})} (\rho_R(\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{x} - [\mathbf{T}_R]\nabla_{\mathbf{X}}\, \delta\mathbf{x})\, dv_R + \int_{\partial v_R(\mathscr{P})} \mathbf{t}_R \cdot \delta\mathbf{x}\, ds_R \int_{s_{cR}(\mathscr{P})} [\delta\mathbf{x} \cdot \mathbf{T}_R]\mathbf{n}_R\, ds_R = 0, \tag{4.1}$$

where $\delta\mathbf{x}$ is an arbitrary field. As before, $\partial v_R(\mathscr{P})$ includes the non-cohesive part of the crack.

Hence, we may select the field

$$\delta\mathbf{x} = \mathbf{F}\mathbf{l}_R, \tag{4.2}$$

where $\mathbf{l}_R$ is an arbitrary but fix unit vector in the reference configuration.

By aid of (2.16 and 2.18) the virtual work at the crack tip for the given variation becomes

$$-\int_{s_{cR}(\mathscr{P})} [\delta\mathbf{x} \cdot \mathbf{T}_R]\mathbf{n}_R\, ds_R = -\mathbf{l}_R \cdot \int_{s_{cR}(\mathscr{P})} [\mathbf{H}^T]\mathbf{T}_R \mathbf{n}_R\, ds_R. \tag{4.3}$$

Introduction of (4.2) and (4.3) into (4.1) accounting for the balance of momentum $(2.15)_1$ yields

$$\mathbf{l}_R \cdot \int_{v_R(\mathscr{P})} ([\mathbf{T}_R]\nabla_\mathbf{X}\mathbf{F} - \mathbf{H}^T(\mathbf{b}-\mathbf{a})\rho_R)\,\mathrm{d}v_R - \mathbf{l}_R \cdot \int_{\partial v_R(\mathscr{P})} \mathbf{H}^T\mathbf{t}_R\,\mathrm{d}s_R + \mathbf{l}_R \cdot \int_{\mathscr{S}_{cR}(\mathscr{P})} [\mathbf{H}^T]\mathbf{T}_R\mathbf{n}_R\,\mathrm{d}s_R = 0. \quad (4.4)$$

Before developing the theory further, we turn to the interpretation of (4.3). Suppose that the possible growth of the crack at time t is to take place along the whole crack front within $\mathscr{P}$ with the velocity

$$\mathbf{w}_R = w(t)\mathbf{l}_R \quad (4.5)$$

in the direction $\mathbf{l}_R$ referred to the undeformed configuration. The surface normal $\mathbf{n}_R$ on $\mathscr{S}_{cR}(\mathscr{P})$ at each particle $\mathbf{X}$ within the cohesive zone and the vector $\mathbf{l}_R$ are supposed to be perpendicular to each other. It is also assumed that the shape of the cohesive zone in the current configuration remains constant during the growth.

Then, the velocity of particles at the surface of the cohesive zone in a common frame is given by

$$\mathbf{v}^{\pm} = -\mathbf{H}^{\pm}\mathbf{w}_R + \mathbf{w} - \mathbf{w}_R, \quad (4.6)$$

where $\mathbf{w}$ is the propagation velocity of the crack front in the spatial configuration. That velocity may differ from $\mathbf{w}_R$ due to a superposed rigid body motion.

The velocity jump appearing in (3.11) becomes

$$[\mathbf{v}] = -w\mathbf{l}_R \cdot [\mathbf{H}^T]. \quad (4.7)$$

On account of (4.7), comparison of (3.9) and (4.3) gives at hand that the left hand side of (4.3) may be interpreted as the rate of change of the surface energy $\Sigma(\mathscr{P})$ per unit length of growth in the direction of $\mathbf{l}_R$.

Since (4.3) is due to a virtual variation, however, it only expresses the energy transformed into surface energy per unit length of virtual growth. Thus, it may be interpreted as a generalized force measure of the forces tending to cause crack growth. Guided by this particular generalized force interpretation of the conditions at a crack tip, we proceed to the definition of a tentative measure which is obtained from the remaining terms of (4.4).

The *apparent crack extension force* in the direction $\mathbf{l}_R$ is defined by

$$\mathscr{f}_{\mathbf{l}_R}(\mathscr{P}) = \mathbf{l}_R \cdot \int_{v_R(\mathscr{P})} ([\mathbf{T}_R]\nabla_\mathbf{X}\mathbf{F} - \mathbf{H}^T(\mathbf{b}-\mathbf{a})\rho_R)\,\mathrm{d}v_R - \mathbf{l}_R \cdot \int_{\partial v_R(\mathscr{P})} \mathbf{H}^T\mathbf{t}_R\,\mathrm{d}s_R. \quad (4.8)$$

In attempt to supersede difficulties due to poor knowledge of the conditions in the vicinity of a crack tip, it is advantageous to investigate circumstances under which the volume integral in (4.8) may be transformed into a surface integral over the boundary ∂v_R of the control volume.

In special cases to be discussed in Section 5 it is possible to introduce an instantaneous potential π at each particle $\mathbf{X}$ in $v_R(\mathscr{P})$ at time t which satisfies

$$\rho_R \nabla_\mathbf{X}\pi(\mathbf{X}) = [\mathbf{T}_R]\nabla_\mathbf{X}\mathbf{F}. \quad (4.9)$$

The instantaneous potential π is a unique and continuous function of $\mathbf{X}$ for each given deformation history up to time t. Thus, the Green transformation applies to the volume integral of $\rho_R \nabla_{\mathbf{X}} \pi$ provided that

$$\rho_R(\mathbf{X}) = \rho_R = \text{const.} \tag{4.10}$$

throughout $v_R(\mathscr{P})$. Since it is assumed in the above interpretation of the force that $\mathbf{l}_R \cdot \mathbf{n}_R = 0$ at the cohesive zone s_{cR}, and since the material properties are continuous on the surface s_R in front of the cohesive zone, the contribution from the integral of $\rho_R \pi \mathbf{n}_R$ over the surface s_R is equal to zero, and (4.8) assumes the form

$$f_{\mathbf{l}_R}(\mathscr{P}) = -\mathbf{l}_R \cdot \int_{v_R(\mathscr{P})} \mathbf{H}^T(\mathbf{b} - \mathbf{a})\rho_R \, \mathrm{d}v_R + \mathbf{l}_R \cdot \int_{\partial v_R(\mathscr{P})} (\rho_R \pi \mathbf{n}_R - \mathbf{H}^T \mathbf{t}_R) \, \mathrm{d}s_R. \tag{4.11}$$

The remaining volume integral can be generally dealt with in two special cases. For quasi-static deformation we put

$$\mathbf{b} = \mathbf{a} = \mathbf{0} \tag{4.12}$$

and (4.11) becomes

$$\begin{aligned} f_{\mathbf{l}_R}(\mathscr{P}) &= \mathbf{l}_R \cdot \int_{\partial v_R(\mathscr{P})} (\rho_R \pi \mathbf{n}_R - \mathbf{H}^T \mathbf{t}_R) \, \mathrm{d}s_R, \\ f_{\mathbf{l}_R}(\mathscr{P}) &= l_R^{\alpha} \int_{\partial v_R(\mathscr{P})} (\rho_R \pi n_{R\alpha} - u^i{}_{;\alpha} t_{Ri}) \, \mathrm{d}s_R \end{aligned} \tag{4.13}$$

in direct notation and component notation respectively. The semicolon denotes total covariant derivative.

The other tractable case is steady-state propagation of the crack. Suppose that the crack propagates under steady-state conditions with the constant velocity $\mathbf{w}_R$ according to (4.5) referred to the undeformed configuration. Then the velocity of each particle apart from a possible superposed rigid body motion is given by the relation

$$\mathbf{v} = -\mathbf{H}\mathbf{w}_R = -w\mathbf{H}\mathbf{l}_R. \tag{4.14}$$

Furthermore, the material time derivative of any sufficiently smooth quantity Ψ defined at a particle $\mathbf{X}$ is given by

$$\dot{\Psi} = -w(\nabla_{\mathbf{X}} \Psi)\mathbf{l}_R. \tag{4.15}$$

Combination of (4.14) and (4.15) readily yeilds

$$\mathbf{l}_R \cdot \mathbf{H}^T \mathbf{a} = \tfrac{1}{2} w^2 \nabla_{\mathbf{X}} (\mathbf{l}_R \cdot \mathbf{H}^T \mathbf{H} \mathbf{l}_R) \mathbf{l}_R. \tag{4.16}$$

On the assumption of $\mathbf{b} = \mathbf{0}$, (4.11) reduces to

$$\begin{aligned} f_{\mathbf{l}_R}(\mathscr{P}) &= \mathbf{l}_R \cdot \int_{\partial v_R(\mathscr{P})} ((\pi + \tfrac{1}{2} w^2 \mathbf{l}_R \cdot \mathbf{H}^T \mathbf{H} \mathbf{l}_R)\rho_R \mathbf{n}_R - \mathbf{H}^T \mathbf{t}_R) \, \mathrm{d}s_R, \\ f_{\mathbf{l}_R}(\mathscr{P}) &= l_R^{\alpha} \int_{\partial v_R(\mathscr{P})} ((\pi + \tfrac{1}{2} w^2 u^i{}_{;\beta} u_{i;\gamma} l_R^{\beta} l_R^{\gamma})\rho_R n_{R\alpha} - u^i{}_{;\alpha} t_{Ri}) \, \mathrm{d}s_R. \end{aligned} \tag{4.17}$$

According to the relation

$$\dot{\rho}_R(\mathbf{X}) = -w(\nabla_{\mathbf{X}} \rho_R) \cdot \mathbf{l}_R = 0, \tag{4.18}$$

it is only necessary to require that ρ_R is constant in the direction $\mathbf{l}_R$ of propagation.

In the case of steady-state propagation of a crack, all requirements justifying the interpretation of (4.3) as the rate of change of the surface energy per unit length of growth are fulfilled. Thus, (4.17) constitutes a balance equation for those parts of the energy that are balancing the surface energy.

The resulting equations (4.13) and (4.17) for the apparent crack extension force in the direction $\mathbf{l}_R$ may be transformed into the spatial configuration by aid of (2.9) and (2.13):

$$\begin{aligned} \mathscr{f}_{\mathbf{l}_R}(\mathscr{P}) &= \mathbf{l}_R \cdot \int_{\partial v(\mathscr{P})} ((\pi + \tfrac{1}{2}w^2 \mathbf{l}_R \cdot \mathbf{H}^T\mathbf{H}\mathbf{l}_R)\rho \mathbf{F}^T\mathbf{n} - \mathbf{H}^T\mathbf{t})\, \mathrm{d}s, \\ \mathscr{f}_{\mathbf{l}_R}(\mathscr{P}) &= l_R{}^\alpha \int_{\partial v(\mathscr{P})} ((\pi + \tfrac{1}{2}w^2 u^i{}_{;\beta} u_{i;\gamma} l_R{}^\beta l_R{}^\gamma)\rho F^j{}_\alpha n_j - u^i{}_{;\alpha} t_i)\, \mathrm{d}s, \end{aligned} \tag{4.19}$$

where $\mathscr{f}_{\mathbf{l}_R}$ for quasi-static deformation is obtained for $w = 0$.

For infinitesimal deformation no distinction has to be made between the deformed and the undeformed configurations of a body. In such cases (4.19) may be written

$$\mathscr{f}_{\mathbf{l}}(\mathscr{P}) = \mathbf{l} \cdot \int_{\partial v(\mathscr{P})} ((\pi + \tfrac{1}{2}w^2 \mathbf{l} \cdot \mathbf{H}^T\mathbf{H}\mathbf{l})\rho \mathbf{n} - \mathbf{H}^T\mathbf{t})\, \mathrm{d}s,$$

$$\mathscr{f}_{\mathbf{l}}(\mathscr{P}) = l^i \int_{\partial v(\mathscr{P})} ((\pi + \tfrac{1}{2}w^2 u^j{}_{,k} u_{j,l} l^k l^l)\rho n_i - u^j{}_{,i} t_j)\, \mathrm{d}s,$$

where a comma denotes covariant derivative.

The main reason for deriving the surface integral representations for the apparent crack extension force is, as already indicated, the relative lack of knowledge of the conditions at crack tips. For prediction of initiation of crack growth the theory may be developed further. Suppose that we know the shape of the fully opened crack but have only rude conceptions of the cohesive zone. Then, we may proceed as follows.

The apparent crack extension force $\mathscr{f}_{\mathbf{l}_R}$ may be computed from any appropriate equation as a function of $\mathbf{l}_R$. Then the unit vector $\mathbf{l}_R$ for which $\mathscr{f}_{\mathbf{l}_R}$ reaches its maximum value may be determined. As a *hypothesis* this direction may be postulated as the direction of preferred crack growth. Growth in this direction will occur when the corresponding force reaches a critical value $\mathscr{f}_c$.

In finite element applications still another interpretation may be utilized. Ahead of the crack tip the elements may be arranged in a fan-shaped configuration with the elements located in one narrow sector modeling the cohesive zone. In this case the direction $\mathbf{l}_R$ of the cohesive zone will be well-defined. On comparison of different directions $\mathbf{l}_R$ for a given load on the body, that giving maximum value of the apparent crack extension force can be determined. The direction thus obtained may be considered, by *hypothesis*, to be the direction of preferred growth of the crack.

We remark, in passing, that a force vector interpretation of the vector measure that is obtained by omitting $\mathbf{l}_R$ in the expressions for the apparent crack extension force is questionable, because the vector measure would be defined in the reference configuration instead of in the actual configuration of the body. Rather than being considered as a force acting in the direction of preferred growth of a crack, the apparent crack extension force $\mathscr{f}_{\mathbf{l}_R}$ should be regarded as a generalized force associated with growth in the direction $\mathbf{l}_R$.

The apparent crack extension force at singular crack borders

Concentrating on singular crack borders, we assume that the displacement gradient exhibits singular behaviour according to

$$\begin{aligned} \mathbf{H} &= 0(R^{\alpha}), \qquad -1 < \alpha < 0, \\ \mathbf{H}\mathbf{l}_{tR} &= 0(1), \end{aligned} \tag{4.21}$$

where R is the distance in the reference configuration between the considered particle and the singular line ℓ_R, and $\mathbf{l}_{tR}$ is the unit tangent vector of ℓ_R in the normal plane of the line that contains the considered particle. The orders of magnitude refer to the Euclidean norm of the functions.

The second condition $(4.21)_2$ is necessary for a one-to-one mapping of the singular line ℓ_R onto ℓ in the spatial configuration, whereas the first condition is only sufficient.

In presence of such singularities regarded functions are not defined on the singular line. Accordingly, the Green transformation does not apply to (4.8). However, we may define the apparent crack extension force at the part $\ell_R(\mathscr{P})$ of a singular crack border contained in the part $\mathscr{P}$ of a body through (4.13) and (4.17) with the surface of integration chosen as the limit surface

$$\partial\ell_R(\mathscr{P}) = \lim_{D\to 0} \mathscr{s}_R(\ell_R(\mathscr{P})), \tag{4.22}$$

where $\mathscr{s}_R(\ell_R(\mathscr{P}))$ is a smooth surface that surrounds the part $\ell_R(\mathscr{P})$ of the singular line and D is an upper bound for the distance between particles located upon that surface and the singular line.

Since the Green transformation applies to any region which does not contain a singular crack border, it is readily shown that the formulae for $\mathscr{f}_{\mathbf{l}_R}$ thus obtained are independent of the choice of integration surface as long as the surface completely encloses the crack border. That is, the integration surface should be homotopic to that defined by (4.22).

The apparent crack extension force may be computed per unit length of the border at any point $\mathbf{X}$ on a smooth singular border line through the line integral

$$\mathscr{f}_{\mathbf{l}_{nR}}(\mathbf{X}) = \mathbf{l}_{nR} \cdot \oint_{\partial\ell_R(\mathbf{X})} (\varepsilon\rho_R\,\mathbf{n}_R - \mathbf{H}^T\mathbf{t}_R)\,\mathrm{d}c_R \tag{4.23}$$

in the case of quasi-static deformation.

Here, $\mathrm{d}c_R$ is the arc element of the integration path $\partial\ell_R(\mathbf{X})$ defined by

$$\partial\ell_R(\mathbf{X}) = \lim_{D\to 0} c_R(\mathbf{X}), \tag{4.24}$$

where $c_R(\mathbf{X})$ is a closed path surrounding the singular line in the normal plane of ℓ_R at $\mathbf{X}$ in which plane also $\mathbf{n}_R$ is situated.

Further, the arbitrary unit vector $\mathbf{l}_{nR}$ has been restricted to the normal plane of the crack border at $\mathbf{X}$ which is due to the condition $(4.21)_2$ and to the fact that $\mathbf{n}_R$ is perpendicular to the tangent of the singular line.

In the fracture hypothesis proposed it may be assumed that the critical value of the apparent crack extension force per unit length of the border as given by (4.23) is a material parameter possibly depending on the temperature.

5. CONSTITUTIVE THEORIES FOR THE MATERIAL

So far we have not discussed the problem of establishing the instantaneous potential π for a given material. This potential with the desired property (4.9) is generally expressible only for special classes of material under certain specified conditions.

Elastic materials

For a homogeneous elastic material there exists a global free energy function ψ which may be expressed as a unique function of the current value of the deformation gradient $\mathbf{F}$, i.e. $\psi = \hat{\psi}(\mathbf{F})$. The first Piola–Kirchhoff stress tensor is given by

$$\mathbf{T}_R = \rho_R \nabla_{\mathbf{F}} \hat{\psi}. \tag{5.1}$$

Introduction of (5.1) into (4.9) shows that $\hat{\psi}$ may be chosen for π.

In the case of homogeneous thermo–elastic materials, the free energy function is given by $\psi = \tilde{\psi}(\mathbf{F}, \vartheta)$ where ϑ is the temperature. The material gradient of ψ is

$$\nabla_{\mathbf{X}} \psi = [\nabla_{\mathbf{F}} \tilde{\psi}] \nabla_{\mathbf{X}} \mathbf{F} + \frac{\partial \tilde{\psi}}{\partial \vartheta} \nabla_{\mathbf{X}} \vartheta. \tag{5.2}$$

Thus, $\tilde{\psi}$ is appropriate for π only under isothermal conditions.

As far as infinitesimal deformation of linearily elastic materials is concerned, $\rho_R \pi$ may be written in terms of stress and strain. Irrespective of the symmetry properties of a given material its constitutive equation is of the form

$$\mathbf{T} = \mathbf{C}[\tilde{\mathbf{E}}], \tag{5.3}$$

where $\mathbf{C}$ is the fourth-order elasticity tensor.

The instantaneous potential becomes

$$\rho_R \pi = \tfrac{1}{2} \mathbf{C}[\tilde{\mathbf{E}} \otimes \tilde{\mathbf{E}}] = \tfrac{1}{2} \mathbf{T}[\tilde{\mathbf{E}}], \tag{5.4}$$

where $\otimes$ denotes tensor product.

Viscoelastic materials

Materials with fading memory, or viscoelastic materials, are materials for which the Helmholtz free energy is given by a constitutive functional

$$\psi(t) = \underset{s \to 0}{\overset{\infty}{\psi}} (\mathbf{F}^t(s), \vartheta^t(s)) \tag{5.5}$$

determined by the total history $(\mathbf{F}^t, \vartheta^t)$ of the deformation gradient and the temperature where

$$\left.\begin{aligned} \mathbf{F}^t(s) &= \mathbf{F}(t-s) \\ \vartheta^t(s) &= \vartheta(t-s) \end{aligned}\right\} \quad \text{for } 0 \le s < \infty. \tag{5.6}$$

The functional (5.5) obeys the assumption of fading memory which is the assumption that the deformations and temperatures occurring in the distant past have less influence on the current state of the material than those occurring in the recent past.

A general theory of materials with fading memory was given by Coleman[7]. The mathematical interpretation of the assumption of fading memory that was investigated by Coleman yields the constitutive equation for the stress tensor in the form

$$\mathbf{T}_R = \rho_R \, D_{\mathbf{F}} \, \underset{s=0}{\overset{\infty}{\psi}} \, (\mathbf{F}^t, \vartheta^t). \tag{5.7}$$

The stress tensor is given by the gradient of the free energy functional with respect to the current value of the deformation gradient.

It is, in general, not possible to deduce a globally valid instantaneous potential π from (5.7). A special case of interest may, however, be derived directly. Restricting the study to isothermal conditions, we consider a body which has been in equilibrium in a preferred reference state up to time t_0. At that time a constant deformation is applied to the material. The functional (5.5) then reduces to a function and (5.7) becomes

$$\mathbf{T}_R = \rho_R \, \nabla_{\mathbf{F}} \bar{\psi}(\mathbf{F}, t - t_0) \qquad \text{for } t > t_0 \tag{5.8}$$

which corresponds to a time dependent elastic material (see Ref. [8]).

We conclude that under isothermal conditions it is possible to compute the apparent crack extension force at stress relaxation with $\bar{\psi}$ chosen for π. For crack problems stress relaxation may not be a very realistic case. However, two important particular cases may be read off from the derived result: the material response immediately after application of the load, and the equilibrium response reached after long time of relaxation. In the terminology introduced by Leigh[9] the corresponding equations are called the rapid-deformation elastic equation and the slow-deformation elastic equation respectively:

$$\begin{aligned} \mathbf{T}_R &= \rho_R \nabla_{\mathbf{F}} \bar{\psi}(\mathbf{F}, 0), \\ \mathbf{T}_R &= \rho_R \nabla_{\mathbf{F}} \bar{\psi}(\mathbf{F}, \infty). \end{aligned} \tag{5.9}$$

These equations may be considered as the extremes of viscoelastic behaviour and, consequently, they may be of interest in search for bounds for the apparent crack extension force. The critical crack extension force may, of course, be different too for the two extreme behaviours of the material. Analogous to the case of elastic materials, the functions $\bar{\psi}(\mathbf{F}, 0)$ and $\bar{\psi}(\mathbf{F}, \infty)$ may be chosen for π in these cases.

For linearly viscoelastic materials obeying a constitutive equation of the form

$$\mathbf{T} = \int_{-\infty}^{t} \mathbf{G}(t - \tau)[\tilde{\mathbf{E}}(\tau)] \, \mathrm{d}\tau, \tag{5.10}$$

where $\mathbf{G}$ is the fourth-order stress–relaxation tensor, the rapid-deformation elastic equation and the slow-deformation elastic equation assume the form

$$\begin{aligned} \mathbf{T} &= \mathbf{G}(0)[\tilde{\mathbf{E}}], \\ \mathbf{T} &= \mathbf{G}(\infty)[\tilde{\mathbf{E}}], \end{aligned} \tag{5.11}$$

where $\mathbf{G}(0)$ is called the rapid-deformation elasticity tensor and $\mathbf{G}(\infty)$ the slow-deformation elasticity tensor.

Thus, for infinitesimal deformation $\rho_R \pi$ may be chosen according to $(5.4)_2$ with the stress tensor given by the appropriate equation (5.11).

Elastic–plastic materials

Elastic–plastic materials exhibit rate-independent effects which do not enable us to find a general class of deformations for which an instantaneous potential π exists being single-valued function of the deformation gradient. Thus, it is in general not possible to apply the surface integral formulae for the apparent crack extension force to elastic–plastic materials.

For a particular constitutive theory it is possible to proceed a little further. Consider the elastic–plastic theory for small strains with the infinitesimal strain tensor decomposed into one elastic and one plastic part:

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^e + \tilde{\mathbf{E}}^p. \tag{5.12}$$

It is assumed that the stress tensor in every point is given by a global elastic equation of the form (see Ref. [10])

$$\mathbf{T} = \mathbf{C}[\tilde{\mathbf{E}}^e]. \tag{5.13}$$

The apparent crack extension force then takes the form

$$f_1(\mathscr{P}) = \mathbf{l} \cdot \int_{\partial v(\mathscr{P})} (\tfrac{1}{2}\mathbf{T}[\tilde{\mathbf{E}}^e]\mathbf{n} - \mathbf{H}^T\mathbf{t})\,\mathrm{d}s + \mathbf{l} \cdot \int_{v(\mathscr{P})} [\mathbf{T}]\nabla_{\mathbf{X}}\tilde{\mathbf{E}}^p\,\mathrm{d}v \tag{5.14}$$

for quasi-static deformation. The corresponding formula for steady-state propagation is quite similar.

The usefulness of (5.14) and corresponding equation for steady-state propagation is probably restricted to numerical calculations.

To the author it does not seem to be possible to extend the derived theory for elastic–plastic materials to a constitutive theory which includes finite deformation. The possibility to describe elastic–plastic materials with aid of plastic potentials which are unique functions of the deformation gradient remains. Such theories are, however, actually theories for elastic materials and will not be discussed here.

6. PLANE DEFORMATION

In this section the three-dimensional formulae for the apparent crack extension force will be specialized to the case of plane deformation. On that account, consider a body with a Cartesian frame (X, Y, Z) in the reference configuration which is deformed in the X–Y-plane.

The body is of unit thickness in the Z-direction and contains a through crack with the surface normal everywhere parallel to the X–Y-plane as indicated in Fig. 2. The arbitrarily chosen surface ∂v_R enclosing the crack tip has an outward unit normal $\mathbf{n}_R$ everywhere parallel to the X–Y-plane. The curve c_R is the projection of ∂v_R in the X–Y-plane and c_R is oriented

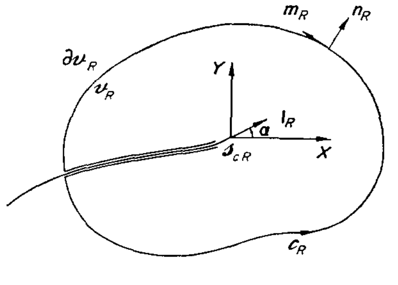

Fig. 2. Integration contour for the apparent crack extension force at plane deformation.

in such a way that upon traversing c_R the volume v_R is to the left. Analogous to the general case, the fully opened crack is included in c_R.

The area-vector $\mathbf{n}_R \, ds_R$ may be written

$$\mathbf{n}_R \, ds_R = \mathbf{m}_R \times (dZ \, \mathbf{i}_Z), \tag{6.1}$$

where $\mathbf{i}_Z$ is the unit vector in the positive Z-direction and $\mathbf{m}_R$ is the positively directed tangent vector of c_R given by

$$\mathbf{m}_R = dX \, \mathbf{i}_X + dY \, \mathbf{i}_Y, \tag{6.2}$$

the signs of dX and dY being suitably chosen.

The possible growth direction $\mathbf{l}_R$ is

$$\mathbf{l}_R = \cos\alpha \, \mathbf{i}_X + \sin\alpha \, \mathbf{i}_Y. \tag{6.3}$$

On the assumption that the displacement vector $\mathbf{u}$ depends explicitly only on the material coordinates, the total covariant derivative $u^i{}_{;\alpha}$ reduces to the partial derivative of u^i with respect to X^α.

Hence, the equations (6.1)–(6.3) yield

$$\begin{aligned} \mathbf{l}_R \cdot \mathbf{n}_R \, ds_R &= (\cos\alpha \, dY - \sin\alpha \, dX) \, dZ, \\ \mathbf{l}_R \cdot \mathbf{H}^T \mathbf{t}_R &= \left(\cos\alpha \frac{\partial u_i}{\partial X} + \sin\alpha \frac{\partial u_i}{\partial Y}\right) t_{Ri}. \end{aligned} \tag{6.4}$$

Let f_α denote the apparent crack extension force in the direction α and c_R the arc length. Then, substitution of (6.4) into (4.13) yields

$$f_\alpha = \int_{c_R} \left(\rho_R \, \pi (\cos\alpha \, dY - \sin\alpha \, dX) - \left(\cos\alpha \frac{\partial u_i}{\partial X} + \sin\alpha \frac{\partial u_i}{\partial Y}\right) t_{Ri} \, dc_R\right). \tag{6.5}$$

The apparent crack extension force f_α may be computed in the spatial frame as well. Let the spatial frame be a Cartesian coordinate system (x, y, z) not necessarily coinciding with the (X, Y, Z) system, but, for the sake of simplicity, the x–y-plane should be the plane of deformation. The surface ∂v enclosing the crack tip is chosen in the same way as ∂v_R with its projection in the x–y-plane denoted by c.

The equations (6.1) and (6.2) should be replaced by similar ones with the subscript R deleted and x, y and z substituted for X, Y and Z, respectively. Then, the equations corresponding to (6.4) read

$$\begin{aligned} \mathbf{l}_R \cdot \mathbf{F}^T \mathbf{n} \, ds &= \left(\cos\alpha \frac{\partial x}{\partial X} + \sin\alpha \frac{\partial x}{\partial Y}\right) dy \, dz - \left(\cos\alpha \frac{\partial y}{\partial X} + \sin\alpha \frac{\partial y}{\partial Y}\right) dx \, dz, \\ \mathbf{l}_R \cdot \mathbf{H}^T \mathbf{t} &= \left(\cos\alpha \frac{\partial u_i}{\partial X} + \sin\alpha \frac{\partial u_i}{\partial Y}\right) t_i. \end{aligned} \tag{6.6}$$

Accordingly, (4.19) with $w = 0$ becomes

$$\begin{aligned} f_\alpha = \int_c \Bigg(\rho\pi\left(\left(\cos\alpha \frac{\partial x}{\partial X} + \sin\alpha \frac{\partial x}{\partial Y}\right) dy - \left(\cos\alpha \frac{\partial y}{\partial X} + \sin\alpha \frac{\partial y}{\partial Y}\right) dx\right) \\ - \left(\cos\alpha \frac{\partial u_i}{\partial X} + \sin\alpha \frac{\partial u_i}{\partial Y}\right) t_i \, dc\Bigg). \end{aligned} \tag{6.7}$$

In the case of infinitesimal deformation, and with coinciding coordinate directions ($\mathbf{i}_x = \mathbf{i}_X$, $\mathbf{i}_y = \mathbf{i}_Y$ and $\mathbf{i}_z = \mathbf{i}_Z$), (6.5) and (6.7) are identical.

On consideration of steady-state propagation of a crack, it is suitable to choose coinciding Cartesian coordinate systems in the two configurations. The crack is assumed to propagate in the positive X-direction with the velocity w. In that case (4.17) becomes

$$\mathscr{f} = \int_{c_R} \left(\left(\pi + \tfrac{1}{2} w^2 \frac{\partial u_i}{\partial X} \frac{\partial u_i}{\partial X} \right) \rho_R \, \mathrm{d}Y - \frac{\partial u_i}{\partial X} t_{Ri} \, \mathrm{d}c_R \right). \tag{6.8}$$

The spatial form of this equation is

$$\mathscr{f} = \int_{c} \left(\left(\pi + \tfrac{1}{2} w^2 \frac{\partial u_i}{\partial X} \frac{\partial u_i}{\partial X} \right) \rho \, \mathrm{d}y - \frac{\partial u_i}{\partial X} t_i \, \mathrm{d}c \right). \tag{6.9}$$

The formulae (6.5) and (6.8) for the apparent crack extension force are valid also for generalized plane deformation, i.e. plane deformation combined with a uniform stretch normal to its plane. If divided by the stretch, (6.7) and (6.9) are likewisely valid. For infinitesimal deformation the validity of the equations obtained is generalized to conditions of plane stress and antiplane strain.

For a crack along the X-axis (6.5) reduces in the case of small strain and with $\alpha = 0$ to the J integral by Rice[1]. The G integral by Sih[2] is included in (6.8).

7. FINAL REMARKS

The proposed fracture theory is based upon the main idea that the cohesive zone of a crack evolves in a favourable direction during loading of a body. Unfortunately, there exists no sufficiently developed continuum mechanical theory which could provide a definite meaning of that favourable direction. Because of its physical interpretation it seems reasonable to the author that the apparent crack extension force could be contemplated as a governing force for crack growth. Being aware of the limitations of the generalized force concept defined, we can only consider the proposed theory as a preliminary one. Nevertheless, it has the advantage of being simple to apply in engineering design. It also includes theories which have proved useful for prediction of initiation of crack growth within their ranges of applicability.

The theory has been applied by Strifors[11] to special fracture problems for isotropic, linearly elastic materials. For instance, an infinite plate with a plane through crack subjected to a uniform tensile load remote from the crack has been considered. The values of the apparent crack extension force and direction of preferred growth have been calculated for different orientations of the crack. On comparison of theoretical predictions with experimental results for polymethyl methacrylate (PMMA), it turns out that the prediction of critical force is good for virtually all crack orientations, whereas the growth direction is satisfactorily predicted only for cracks oriented favourably for growth.

It ought to be emphasized that the apparent crack extension force is a measure of force and in general not a measure of the energy available for extension of cracks. This remains true for J since it is a special case of $\mathscr{f}_{1_R}$. Irwin[3] has pointed out very clearly the generalized force interpretation of the forces tending to cause crack growth. In the case of steady-state propagation of cracks, the distinction between the concepts of crack extension force and energy release rate becomes needless as a consequence of the heavy kinematical demands inherent in the steady-state condition.

An important question which has been discussed in connection with generalized force measures is whether an actual growth will be stable, i.e. controllable by the applied loads, or unstable. Since there is in general, no correlation between the apparent crack extension force and the energy available to supply the surface energy at growth, it is clear that a condition $\mathscr{f}_{1_R\ \max} < \mathscr{f}_c$ only may furnish a sufficient condition for stability.

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Абстракт — В рамках механики сплошной среды определяется предварительная мера усилий, имеющих тенденцию вызывания роста трещины — очевидная сила распространения трещины. На основе присоединенного критерия излома, можно предсказать как возниковение роста, так и направление преимущественного роста. Приспособливается теория к упругим, вязкоупругим и упруго-пластическим материалам. Для заданных условий очевидная сила распространения трещины может быть выражена поверхностными интегралами по границе произволной части тела, для квази-статической деформации и для стационарного распространения трещины. Для плоской деформации и для инфинитезимальной деформации в условиях плоского напряжения эти поверхностные интегралы сокращаются к независимым линейным интегралам по контуре. Они заключают, в качестве специальных случаев, интеграл Райса и интеграл Сига.